BIOT'S VARIATIONAL PRINCIPLE IN TRANSIENT HEAT CONDUCTION WITH NON-LINEAR BOUNDARY CONDITIONS

BY

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AUGUST, 1970

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A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

BY

PARIKSHITA NAYAK

to the

DEPARTMENT OF MECHANICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

AUGUST, 1970

CERTIFICATE

This is to certify that the present work has been carried out under my supervision and the work has not been submitted elsewhere for a degree.

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ACKNOWLEDGEMENT

I am indebted to Dr. H.C. Agrawal for introducing me to this area, for guidance and stimulating discussions during the course of this undertaking.

My thankfulness goes to all my friends, particularly, to Mr. K.N. Srivastav who, generously spared time for useful discussions. Lastly, I thank Shri Uma Raman Pandey for his neat and prompt typing.

PARIKSHITA NAYAK

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of the
Dissertation on

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Transient, one-dimensional temperature distribution is determined for bodies with internal heat generation and non-linear boundary conditions. Approximate analytical solution is obtained with the aid of Biot's variational method. A parabolic profile of temperature distribution is assumed. Results are illustrated graphically and compared with available exact solutions to ascertain the efficacy and accuracy of Biot's method.

NOMENCLATURE

$\mathcal{F}_{se}$ - Hottel's Radiation exchange factor between solid and environment

q - Surface heat flux

h_1 - Heat transfer coefficient

K - Thermal conductivity

L - Thickness of the plate

N_{Bi} - Biot Number, $\frac{h_1 L}{K}$

N_{Rh} - Radiation number, $\frac{\mathcal{F}_{se}\,\sigma\,T_a^3\,L}{K}$

T - Absolute Temperature

T_f - Ambient fluid temperature

T_e - Environment temperature

T_a - Adiabatic surface temperature

T_o - Initial plate temperature

T_s - Surface temperature

t_1 - time

t - Dimensionless time $\frac{\alpha t_1}{L^2}$

U_i - Dimensionless initial temperature

U_s - Dimensionless surface temperature

x_1 - Space co-ordinate

x - Dimensionless space co-ordinate $\frac{x_1}{L}$

α - Thermal diffusivity

σ - Stefan Boltzman constant

CHAPTER I

INTRODUCTION

In the world of physical reality, almost all the problems are non-linear and it is very difficult to solve them exactly because of the complicacy involved in the mathematical analysis.

The problem of transient heat conduction in a solid becomes non-linear when the thermophysical properties are dependent on temperature or the surface heat flux is a non-linear function of temperature. Non-linearity because of the boundary condition, such as radiation heat transfer, is becoming increasingly important with the advent of space exploration and development of areas like cryogenics and plasmas where large temperature differences exist.

The present study is concerned with a non-linear transient heat conduction problem resulting from a non-linear surface flux. In general, an opaque solid is subjected to combined convection and radiation heat fluxes at the surface with internal heat generation. When one mode of energy transfer at the surface predominates the other; two limiting cases arise :

(i) pure thermal radiation, and

(ii) pure convection.

The radiation boundary condition is always non-linear and the convection boundary condition is non-linear except for

the case of forced convection with heat transfer co-efficient independent of surface temperature.

Approximate analytical methods of solution to heat conduction problem have received considerable attention in the past few years. Attention has been drawn to the necessity of including non-linear boundary condition and material property variation into the analysis of heat conduction problems. Since classical methods of heat conduction analysis cannot account for these effects so easily, approximate methods of analysis which can include these two effects in a systematic manner are necessary to be discussed. Two important approximate methods that have been introduced with this need in mind are the heat balance integral method due to Goodman and the variational method by Biot. The importance of Biot's variational method has been well recognised and considerable work has been done in the recent past.

The problem of determination of the temperature distribution in a flange-web combination was treated by Biot[1] as a one-dimensional unsteady problem. An extention of this variational method for the same problem for the two-dimensional case was given by Levinsion[2]. Citron[3-4], in addition, has used Biot's principle with reference to ablation problems. In Citron's work, the variational principle was expressed in terms of a functional expression involving heat flux and the temperature gradient. The variation was carried out with respect to the heat flux while the temperature gradient was held fixed. A more recent application of

the principle to ablation problems was given by Biot and Agrawal[5] where constant surface heat flux condition was used. Lardner[6] has solved various problems using Biot's principle with heat flux boundary conditions. In his analysis, he has evaluated (n-1) of the generalised co-ordinates from the variational principle while the nth co-ordinate has been derived from the overall energy balance. The important practical case of non-linear boundary condition of combined radiation and convection has not so far been treated by the Biot's variational method or any other approximate method. The same has been achieved in the present analysis by solving the problem of transient heat conduction with non-linear boundary condition, the exact solution of which is already available [13, 14, 15].

CHAPTER II

BIOT'S VARIATIONAL PRINCIPLE IN TRANSIENT HEAT CONDUCTION WITH NON-LINEAR BOUNDARY CONDITION

2.1 STATEMENT OF THE PROBLEM

Transient temperature distribution in a plate, initially at a uniform temperature is suddenly subjected to uniform heat generation inside and thermal radiation combined with forced convection heat transfer at the surface. The other surface is kept insulated.

ASSUMPTIONS : -

The following assumptions are made to simplify the problem :

I) Heat conduction is one-dimensional.

II) The plate is isotropic, homogeneous and opaque to thermal radiation.

III) Physical properties are independent of temperature.

IV) The temperature of the environment and that of the fluid forced past the plate are not functions of time.

V) The heat transfer co-efficient is independent of surface temperature.

VI) The fluid is transparent to thermal radiation.

The plate is of thickness L in x_1-direction and extends to infinity in the other two directions so that the conduction is essentially one-dimensional.

The initial temperature of the plate has been taken as uniform. The plate is suddenly placed in contact with a large mass of fluid at temperature T_f, which is forced across the surface of the plate, and is also surrounded by an enclosure at a temperature T_e with which there is radiative exchange. Heat generation at a constant and uniform rate starts simultaneously in the plate. The plate is insulated on one face. It is required to find the temperature distribution in the plate under the above conditions and the surface temperature at any instant.

2.2 FORMULATION OF THE PROBLEM

The equation for transient temperation distribution in the plate with heat generation is

$$\frac{\partial T(x_1, t_1)}{\partial t_1} = \alpha \frac{\partial^2 T(x_1, t_1)}{\partial x_1^2} + \frac{U'''}{\rho C_p} \qquad (1)$$

where
α = Diffusivity
= $K/\rho C_p$
ρ = Density
C_p = Specific heat at constant pressure
U''' = Constant rate of heat generation per unit volume.

Assuming the initial temperature distribution in the plate to be constant, the initial condition is

$$T(x_1, 0) = T_o \qquad (2a)$$

Since the plate is insulated at $x_1=0$, the temperature

gradient at this end is zero

$$\frac{\partial T}{\partial x_1} = 0 \qquad \text{at} \qquad x_1 = 0 \tag{2b}$$

The final boundary condition is just a statement of an energy balance at the surface in contact with the fluid. This gives

$$q\,(T_s) = -K\,\frac{\partial T}{\partial x_1}$$

$$= \mathcal{F}_{se} \cdot \sigma\,(T_s^4 - T_e^4) + h_1(T_s - T_f) \tag{2c}$$

The left hand side of equation (2c) represents heat transfer by conduction, while the first term on the right hand side accounts for radiation and the second term for convection of heat. In writing this boundary condition, it has been assumed that the surface of the plate is gray and diffuse and the radiation incident on the surface is uniform over the surface.

The boundary condition can be written in a more managable form by defining an adiabatic temperature T_a, at which the surface flux, $q(T_a)$ is zero.

Thus,

$$q(T_a) = 0 = \sigma \cdot \mathcal{F}_{se}(T_a^4 - T_e^4) + h_1(T_a - T_f) \tag{2d}$$

$$\therefore\ \sigma \cdot \mathcal{F}_{se}\, T_e^4 + h_1 T_f = \sigma \cdot \mathcal{F}_{se}\, T_a^4 + h_1 T_a \tag{2e}$$

The surface flux is written in terms of the adiabatic surface temperature as

$$q(T_s) = \mathcal{F}_{se}\,\sigma\,(T_s^N - T_a^N) + h_1(T_s - T_a) \tag{2f}$$

The power N indicates a very general behaviour of the radiation condition. It is to be noted that a general case in which the fluid temperature differs from the environment temperature is included in the analysis.

2.3 NON-DIMENSIONALISATION

To simplify the analysis, the basic equation (1) is written in terms of the following dimensionless variables :

Length $\quad x = \dfrac{x_1}{L}$

Time $\quad t = \dfrac{\alpha t_1}{L^2}$

Temperature $U(x,t) = \dfrac{T(x_1,t_1)}{T_a}$ and the surface temperature at $x_1=L$ is given by

$$U_s(t) = U(1,t) = \frac{T_s}{T_a}$$

Then

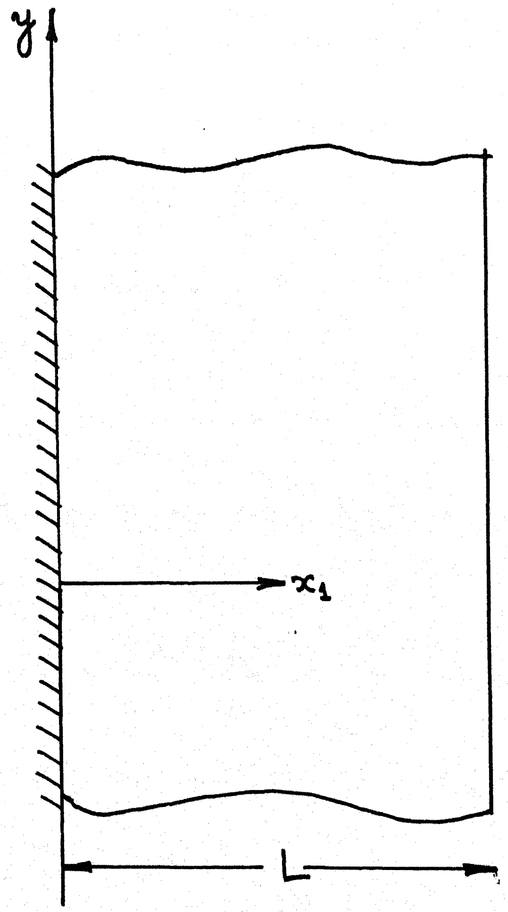
$$\frac{\partial T(x_1,t_1)}{\partial t_1} = T_a \cdot \left(\frac{\partial U}{\partial t}\right)\left(\frac{\partial t}{\partial t_1}\right)$$

$$= T_a \cdot \frac{\alpha}{L^2} \cdot \frac{\partial U}{\partial t}$$

Also

$$\frac{\partial T(x_1,t_1)}{\partial x_1} = T_a \left(\frac{\partial U}{\partial x}\right)\left(\frac{\partial x}{\partial x_1}\right)$$

$$= \frac{T_a}{L} \cdot \frac{\partial U}{\partial x}$$



y
x_1
L

$$\therefore \quad \frac{\partial^2 T(x_1,t_1)}{\partial x_1^2} = \frac{T_a}{L} \; \frac{\partial^2 U}{\partial x^2} \cdot \frac{\partial x}{\partial x_1}$$

$$= \frac{T_a}{L^2} \cdot \frac{\partial^2 U}{\partial x^2}$$

Substituting these expressions in equation (1) we have

$$T_a \cdot \left(\frac{\alpha}{L^2}\right) \cdot \frac{\partial U}{\partial t} = \frac{T_a}{L^2} \; \frac{\partial^2 U}{\partial x^2} + \frac{u'''}{C_p}$$

or $$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{u''' L^2}{K T_a}$$

or $$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + g''' \qquad (3)$$

where $g''' = \frac{u''' L^2}{K T_a}$ = the dimensionless heat generation.

The initial condition (2a) may be non-dimensionalised in a similar manner

$$U(x,0) = \frac{T_0(x_1,0)}{T_a} = U_i(x) \qquad (4a)$$

The boundary condition (2b) becomes

$$U_x(0,t) = 0 \qquad (4b)$$

Now the boundary condition (2c) takes the form

$$-\frac{K}{T_a} \cdot \frac{T_a}{L} \; \frac{\partial U(1,t)}{\partial x}$$

$$= \frac{\sigma \cdot F ea}{T_a} (T_s^4 - T_a^4) + \frac{h_1}{T_a} (T_s - T_a)$$

$$\text{or} \quad \frac{\partial U(1,t)}{\partial x} = -\left[\frac{\sigma \varepsilon L}{K} \cdot T_a^3 (U_s^N - 1) + \frac{h_c L}{K} (U_s - 1) \right]$$

$$= -\left[N_{Rh} (U_s^N - 1) + N_{Bi} (U_s - 1) \right] \quad \ldots\ldots (4)$$

where N_{Rh} and N_{Bi} are dimensionless radiation number and Biot number respectively. Our aim is to find the transient temperature distribution $U(x,t)$ as well as the surface temperature U_s. We will solve the problem by Biot's variational principle. The use of this variational principle requires the particular form of temperature profile and a sufficient number of generalised co-ordinates applicable to the thermal system. Experience of previous workers have shown that a parabolic profile is a suitable choice with regard to both convenience and accuracy.

2.4 BIOT'S VARIATIONAL PRINCIPLE

Although the variational principle for heat transfer introduced by Biot[1] is based on the theory of irreversible thermodynamics, only, the mathematical formulation will be reviewed in this section.

The variational principle is stated in terms of a heat flow vector field $\vec{H}$, whose time rate of change $\dot{\vec{H}}$ is the heat flux across an area normal to $\dot{\vec{H}}$. Since energy must be conserved in the heat transfer process, it is necessary that

$$\int \vec{H} \cdot \vec{n} \, dS = -\int C \, \theta \, dv$$

$$\text{or} \quad \nabla \cdot \vec{H} = -C\theta \qquad (5)$$

where θ is the temperature field above an equilibrium temperature, C is the specific heat per unit volume and H is the heat flow vector field. The variational principle is stated as

$$\delta V + \delta D = -\int \theta \, \delta \vec{H} \cdot \vec{n} \, ds \qquad (6)$$

where

$$V = \int \frac{1}{2} C \, \theta^2 \, dv \qquad (7a)$$

and $$\delta D = \int \left(\frac{1}{K}\right) (\dot{\vec{H}}) \cdot \delta \vec{H} \, dv \qquad (7b)$$

The function V is the thermal potential and is related to the thermal energy of the system while the dissipation function D is equal to the production of entropy of the system. For arbitrary variations in the heat flow field, the variational principle can be shown to be equivalent to the heat conduction equation.

$$\nabla \cdot (K \nabla \theta) = C \frac{\partial \theta}{\partial t} \qquad (8)$$

The variation in the thermal potential V yields

$$\delta V = \int C \, \theta \, \delta \theta, \, dv \qquad (9)$$

Using energy conservation equation (5) we get

$$\delta V = -\int \theta \, \delta \vec{H} \cdot \vec{n} \, ds + \int \delta \vec{H} \cdot \nabla \theta dv \qquad (10)$$

The variational principle then leads to the following equation,

$$\int \left(\nabla \theta + \frac{1}{K} \dot{\vec{H}} \right) \cdot \delta \vec{H} \, . \, dv = 0 \qquad (11)$$

For arbitrary variations $\delta \vec{H}$, this reduces to

$$K \nabla \theta + \dot{\vec{H}} = 0 \qquad (12)$$

which on taking divergence leads to the energy equation (8). Therefore, the variational principle given by equation (6) which is to be satisfied for arbitrary variations in the heat flow field is equivalent to equation (12). This result is Fourier's law of conduction relating the heat flux $\dot{\vec{H}}$ to the temperature gradient. Equation (12) with equation (5) yields the heat conduction equation (2) and the equivalence of the variational principle to the heat conduction equation is thus established.

In any approximate solution employing the variational principle Fourier's law of conduction is approximated and not the conservation of energy. Thus the fundamental requirement of energy conservation is enforced. Equivalently we can say that any variations in the temperature field are constrained by variations in the heat flow field.

The advantage of Biot's variational principle is that it can be stated in terms of equations of the Lagrangian type for generalised co-ordinates defining the heat flow field. The heat flow vector field can be considered as a function of n generalised co-ordinates $q_n(t)$ in the form

$$H = H(q_n, x, y, z, t)$$

where the generalised co-ordinates specify the configurations of the heat flow field as a function of time. The variational principle can be written in terms of the

generalised co-ordinates in the form

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \tag{13}$$

where the dissipation function

$$D = \int \left(\frac{1}{2K} \right) (\dot{H})^2 \, dv$$

and the generalised thermal force

$$Q_i = \int \theta \left(\frac{\partial H_n}{\partial q_i} \right) ds$$

The Lagrangian form of the equation for the generalised co-ordinates was established by Biot[1,7,8,9] on the basis of linear irreversible thermodynamics. However, since the formulation of variational principle leads to the standard form of heat conduction equation, which is known to hold for large temperature variation, the variational principle has wider application than the thermodynamical arguments would suggest. That is, the range of validity of the variational formulation is equivalent to the range of heat conduction equation.

Now the energy equation for transient heat conduction with internal heat generation takes the form

$$\nabla \cdot (K \nabla \theta) = c \frac{\partial \theta}{\partial t} - U''' \tag{14}$$

where $U''' = U'''(t)$ is the heat generation per unit volume per unit time. In order to evaluate the temperature field we shall modify the existing Biot's principle as follows :

Define the thermal flow field by a vector field

$$\vec{H} = H^{*} + H^{*} \tag{15}$$

where

$$\nabla \cdot H^{*} = - C\,\theta \tag{16}$$

$$\text{and} \quad \nabla \cdot H^{*} = \int_0^t U''' \, dt \tag{17}$$

We express the temperature θ in the form

$$\theta = \sum_i q_i(t) \cdot \theta_i(x) \tag{18}$$

where $\theta_i(x)$ is a function of space and $q_i(t)$ is a generalised co-ordinate dependent on time. Now the unknown portion of the heat flow vector is

$$H^{*} = \sum_i q_i(t) \cdot H_i^{*}(x) \tag{19}$$

The variational principle, with the same expressions for dissipation function and thermal potential as defined earlier can be written in the Lagragian form as

$$\frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \; , \; i = 1 \, \ldots\ldots \, n \tag{20}$$

From the set of equations (20), we evaluate the dependence of generalised co-ordinates q_i on time. The temperature history will be obtained by introducing these co-ordinates into the assumed form of temperature field.

CHAPTER III

SOLUTION OF THE PROBLEM USING BIOT'S METHOD

Let us assume that at $t = 0$ there appears a homogeneous internal heat generation

$$U''' = U'''(t) \tag{21}$$

The solid of initial temperature $U(x,0) = U_i$, exchanges heat energy by convection and radiation with environment the temperature of which is U_i = const. Let us assume the non-dimensional temperature field in the parabolic form as

$$U = (q_1 - q_2)\, x^2 + q_2 \tag{22}$$

where $q_1 = q_1(t)$ and $q_2 = q_2(t)$ are unknown dependents on time generalised co-ordinates. They correspond to instant temperatures of the surface at $x = 1.0$ and $x = 0$, respectively. Let the equation satisfy the boundary condition (4c)

$$U_x(1,t) = -\left[N_{Rh}\left(U_s^N(t)\Big|_{x=1} - 1\right) + N_{Bi}\left(U_s(t)\Big|_{x=1} - 1\right) \right]$$

or $$2(q_1 - q_2) = -\left[N_{Rh}(q_1^N - 1) + N_{Bi}(q_1 - 1) \right]$$

or $$2(q_1 - q_2) + N_{Rh} \cdot q_1^N + N_{Bi} \cdot q_1 - h = 0 \tag{23}$$

where $h = (N_{Rh} + N_{Bi})$

For one-dimensional problem, we have

$$\nabla . \vec{H} = \frac{\partial \vec{H}}{\partial x_1} \tag{24a}$$

or $$\nabla . (H^* + H^*) = \frac{\partial \vec{H}}{\partial x_1} . \frac{\partial x}{\partial x_1}$$

or $$- CU\, T_a + \int_0^{t_1} U''' \, dt_1 = \frac{\partial \vec{H}}{\partial x} . \frac{1}{L}$$

or $$- U + \int_0^{t} g''' \, dt = \frac{1}{L.C.T_a} . \frac{\partial \vec{H}}{\partial x}$$

$$= \frac{\partial \bar{H}}{\partial x} \tag{24b}$$

where $\bar{H} = \frac{H}{L.C.T_a}$ is the non-dimensional heat flow vector.

Integrating equation (24b) we get

$$\bar{H} = \left\{ - \int U \, dx + \int \left[\int_0^{t} g''' \, dt \right] dx + \text{Const.} \right\} . \vec{e}$$

or $$\bar{H} = \left\{ - \int \left[(q_1 - q_2) x^2 + q_2 \right] dx + x \int_0^{t} g''' \, dt. + \text{const.} \right\} . \vec{e}$$

$$= \left\{ - \left[(q_1 - q_2) \frac{x^3}{3} + q_2 x \right] + x \int_0^{t} g''' \, dt \right\} . \vec{e} \tag{25}$$

The condition $\bar{H}$ (0,t) at x=0 has been used and $\vec{e}$ denotes the unit vector directed along the co-ordinate x.

Differentiating equation (25) with respect to time we get

$$\dot{H}(x,t) = \left\{ - \left[(\dot{q}_1 - \dot{q}_2) \frac{x^3}{3} + \dot{q}_2 \, x \right] + x \cdot g''' \right\} \cdot \bar{e} \qquad (26)$$

The thermal potential V is given by

$$V = \frac{1}{2} \int C \, \theta^2 \, dv$$

$$= \frac{1}{2} \int_0^1 C \, T_a^2 \cdot U^2 \cdot L \, dx$$

$$= \frac{C \cdot T_a^2 L}{2} \cdot \int_0^1 U^2 \, dx \qquad (27a)$$

Defining a non-dimensional thermal potential

$$\bar{V} = \frac{V}{C \cdot T_a^2 \cdot L}$$

we get

$$\bar{V} = \frac{1}{2} \int_0^1 U^2 \, dx$$

$$= \frac{1}{2} \int_0^1 \left[(q_1 - q_2) \, x^2 + q_2 \right]^2 dx$$

$$= \frac{1}{2} \left[(q_1 - q_2)^2 \frac{x^5}{5} + q_2^2 \, x + 2 q_2 (q_1 - q_2) \frac{x^3}{3} \Big|_0^1 \right]$$

$$= \frac{1}{30} \left[3 \, q_1^2 + 4 q_1 q_2 + 8 q_2^2 \right] \qquad (27b)$$

Dissipation function D is given by

$$D = \frac{1}{2K} \cdot \int (\dot{H})^2 \, dv$$

$$= \frac{1}{2K} \int (f^* + f^{\circ})^2 \, dx \cdot L$$

$$= \frac{L}{2K} \int \left[(f^*)^2 + (f^{\circ})^2 + 2 f^* \cdot f^{\circ} \right] dx \qquad (28a)$$

Defining a non-dimensional dissipation function

$$\overline{D} = \frac{D \cdot K}{L}$$

we get equation (28a) as

$$\overline{D} = D_1 + D_2 + D_3 \tag{28b}$$

where

$$D_1 = \frac{1}{2} \int_0^1 (f'')^2 \, dx$$

$$= \frac{1}{2} \int_0^1 x^2 \left[(\dot{q}_1 - \dot{q}_2) \frac{x^2}{3} + \dot{q}_2 \right]^2 dx$$

$$= \frac{1}{2} \left[(\dot{q}_1 - \dot{q}_2)^2 \frac{x^7}{7 \times 9} \dot{q}_2^2 \frac{x^3}{3} + \frac{2}{3} \dot{q}_2 (\dot{q}_1 - \dot{q}_2) \cdot \frac{x^5}{5} \right]_0^1$$

$$= \frac{1}{2} \left[(\dot{q}_1 - \dot{q}_2)^2 \frac{1}{63} + \frac{\dot{q}_2^2}{3} + \frac{2}{15} \dot{q}_2 (\dot{q}_1 - \dot{q}_2) \right]$$

$$= \frac{1}{630} \left[5 \dot{q}_1^2 + 32 \dot{q}_1 \dot{q}_2 + 68 \dot{q}_2^2 \right] \tag{29a}$$

$$D_2 = \frac{1}{2} \int_0^1 (f'')^2 \, dx$$

$$= \frac{1}{2} \int_0^1 (g''' x)^2 dx = \left[g''^2 \frac{x^3}{6} \right]_0^1$$

$$= \frac{g''^2}{6} \tag{29b}$$

and $$D_3 = \int_0^1 \left[- x \left\{ (\dot{q}_1 - \dot{q}_2)\frac{x^3}{3} + \dot{q}_2 \right\} x\, g''' \right] dx$$

$$= - \int_0^1 \left[(\dot{q}_1 - \dot{q}_2)\, g'''\, \frac{x^4}{3} + g'''\, \dot{q}_2\, x^2 \right] dx$$

$$= - \left[(\dot{q}_1 - \dot{q}_2)\, g'''\, \frac{x^5}{15} + g'''\, \dot{q}_2\, \frac{x^3}{3} \right]_0^1$$

$$= - \left[(\dot{q}_1 - \dot{q}_2)\, g''' \cdot \frac{1}{15} + g''' \frac{\dot{q}_2}{3} \right]$$

$$= - \frac{1}{15} \left[4\, \dot{q}_2 + \dot{q}_1 \right] g''' \qquad (29c)$$

Adding equations (29a), (29b), and (29c) we get finally the required dissipation function

$$\bar{D} = \frac{1}{630}\left[5\, \dot{q}_1^2 + 32\, \dot{q}_1 \dot{q}_2 + 68\, \dot{q}_2^2 \right] + \frac{g'''}{6} - \frac{1}{15}\left[4\dot{q}_2 + \dot{q}_1 \right] g''' \qquad (30)$$

Differentiating equation (30) with respect to $\dot{q}_1$ we get

$$\frac{\partial \bar{D}}{\partial \dot{q}_1} = \frac{1}{630}\left[10\, \dot{q}_1 + 32\, \dot{q}_2 \right] - \frac{g'''}{15} \qquad (31)$$

Also differentiating equation (27b) with respect to q_1, we get

$$\frac{\partial V}{\partial q_1} = \frac{1}{30}\left[6\, q_1 + 4\, q_2 \right] \qquad (32)$$

From the definition we can get thermal force Q_1 as follows

$$Q_1 \cdot \delta q_1 = U\Big|_{x=1} \cdot \delta H\Big|_{x=1}$$

or $\quad Q_1 \cdot \delta q_1 = q_1 \cdot \frac{\delta q_1}{3}$

or $\quad Q_1 = \frac{q_1}{3}$ $\qquad$ (33)

Introducing equations (31), (32) and (33) in the Lagrangian equation (20) we obtain the algebraic equation to solve for the co-ordinate q_1.

Thus

$$\frac{1}{30}\left[6\, q_1 + 4\, q_2\right] + \frac{1}{630}\left[10\, \dot{q}_1 + 32\, \dot{q}_2\right] - \frac{g'''}{15} = \frac{q_1}{3} \qquad (34)$$

Cancelling common terms and re-arranging, we get

$$42\,(q_2 - q_1) + (5\dot{q}_1 + 16\, \dot{q}_2) - 21\, g''' = 0 \qquad (35)$$

Using the boundary condition (28), we get

$$q_2 = q_1\left[1 + \frac{1}{2}\,(H_{1h} \cdot q_1^{N-1} + H_{1i})\right] - \frac{h}{2} \qquad (36)$$

Differentiating equation (36) with respect to time we get

$$\dot{q}_2 = \dot{q}_1\left[1 + \frac{1}{2}\,(H_{1h}\, q_1^{N-1} + H_{1i})\right] + q_1\left[\frac{N-1}{2} \cdot H_{1h}\, q_1^{N-2} \cdot \dot{q}_1\right]$$

$$= \dot{q}_1\left[1 + \frac{N}{2} \cdot H_{1h} \cdot q_1^{N-1} + \frac{H_{1i}}{2}\right] \qquad (37)$$

Plugging these values of q_2 and $\dot{q}_2$ in the Lagrangian equation (34) we get

$$42\left[q_1\,(1 + \frac{1}{2}\,(H_{1h} \cdot q_1^{N-1} + H_{1i})) - \frac{h}{2} - q_1\right] + 5\dot{q}_1 + 16\Big[$$

$$\dot{q}_1 \left(1 + N\cdot \frac{N_{Mh}}{2} \cdot q_1^{N-1} + \frac{N_{Bi}}{2}\right)\Big] - 21\, g''' = 0$$

or
$$21 \left[N_{Mh}\cdot q_1^{N} + q_1\cdot N_{Bi} - h\right] + \Big[21 + 2\cdot N\cdot N_{Mh}\cdot q_1^{N-1}$$

$$+ 2\, N_{Bi}\Big]\, \dot{q}_1 - 21\, g''' = 0 \qquad (38)$$

It is to be noted here that the internal heat generation term may be of any function of time, but we will consider only the step function

$$g''' = g_0'''\, \eta(t) \qquad (39)$$

where

$$\eta(t) = 0 \quad \text{for} \quad t < 0$$
$$\;\;\;\;\;\;\;\; = 1 \quad \text{for} \quad t \geq 0$$

In this case the equation (38) can be written as

$$21 \left[N_{Mh}\cdot q_1^{N} + N_{Bi}\cdot q_1 - h - g_0'''\right]$$

$$+ \left[21 + 2N \cdot N_{Mh} \cdot q_1^{N-1} + 2\, N_{Bi}\right] \frac{dq_1}{dt} = 0$$

or

$$\frac{N\, q_1^{N-1} + \dfrac{N_{Bi}}{N_{Mh}} + \dfrac{21}{2N_{Mh}}}{q_1^{N} + \dfrac{N_{Bi}}{N_{Mh}}\cdot q_1 - \dfrac{h\cdot g_0'''}{N_{Mh}}} = -\frac{21}{2}\, dt$$

Integrating we get

$$I_1 + I_2 = -\frac{21}{2}\, t \qquad (40)$$

where

$$I_1 = \int_{U_1}^{U_s} \frac{N\, q_1^{N-1} + \dfrac{N_{Bi}}{N_{Rh}}}{q_1^N + \dfrac{N_{Bi}}{N_{Rh}} \cdot q_1 - \dfrac{h + g_0'''}{N_{Rh}}} \cdot d\, q_1$$

$$= \int_{U_1}^{U_s} \frac{N\, q_1^{N-1} + A}{q_1^N + A\, q_1 - B} \qquad (41)$$

and

$$I_2 = \int_{U_1}^{U_s} \frac{\left(\dfrac{2l}{S\, N_{Rh}}\right)}{q_1^N + \dfrac{N_{Bi}}{N_{Rh}} \cdot q_1 - \dfrac{h + g_0'''}{N_{Rh}}} \cdot d\, q_1$$

$$= \lambda \int_{U_1}^{U_s} \frac{d\, q_1}{q_1^N + A\, q_1 - B} \qquad (42)$$

$$A = \frac{N_{Bi}}{N_{Rh}}$$

$$B = \frac{h + g_0'''}{N_{Rh}}$$

and $\lambda = \dfrac{2l}{S \cdot N_{Rh}}$

The equation (40), thus obtained, represents time as a function of surface temperature. The temperature distribution and the surface temperature of the plate will be

presented graphically as functions of time, for different cases discussed below.

CASE I (Combined Radiation and Convection Boundary Condition)

Consider values of Radiation number (N_{Rh}) and Biot number (N_{Bi}) finite and greater than zero, and let the initial temperature $U_i \neq 0$. The first integral I_1 (equation 41) is rewritten as

$$I_1 = \int_{U_i}^{U_s} \frac{N\, q_1^{N-1} + A}{q_1^N + A\, q_1 - B} \cdot dq_1$$

This can be easily integrated by direct methods and gives

$$I_1 = L_n \left[q_1^N + A\, q_1 - B \right]_{U_i}^{U_s}$$

If we consider radiation heat transfer for the surface according to Stefan-Boltzman law, the power $N = 4.0$

$$\therefore \quad I_1 = L_n \left[q_1^4 + A\, q_1 - B \right]_{U_i}^{U_s}$$

$$= L_n \frac{B - AU_s - U_s^4}{B - AU_i - U_i^4} \qquad (43)$$

Considering the second integral I_2 (equation 42) we have

$$I_2 = \lambda \int_{U_i}^{U_s} \frac{dq_1}{q_1^N + Aq_1 - B}$$

Using again the Stefan-Boltzman law of radiation, I_2 is simplified to

$$I_2 = \lambda \int_{U_1}^{U_s} \frac{dq_1}{q_1^4 + Aq_1 - B}$$

In order to evaluate this integral we must simplify the denominator

$$q_1^4 + A\,q_1 - B \qquad (44)$$

Expression (44) can be expressed as

$$\left(q_1^2 + \frac{y}{2}\right)^2 - y\,q_1^2 + A\,q_1 - \left(\frac{y^2}{4} + B\right) \qquad (45)$$

where $y \neq 0$ is any real root of the canonical equation

$$y^3 + 4By - A^2 = 0 \qquad (46)$$

Using Cardano's formula (Appendix II) we obtain the real root of the equation (46) as

$$y = \sqrt[3]{\frac{A^2}{2} + \sqrt{\left(\frac{A^2}{2}\right)^2 + \left(\frac{4B}{3}\right)^3}} + \sqrt[3]{\frac{A^2}{2} - \sqrt{\left(\frac{A^2}{2}\right)^2 + \left(\frac{4B}{3}\right)^3}} \qquad (47)$$

For any value of A and B we have $y \neq 0$.

From equation (46), by solving, we can get B interms of A and y as

$$B = -\left(\frac{A^2 - y^3}{4y}\right) \qquad (48)$$

Now we can break-up the expression (44) as

$$(q_1^2 + a_1 q_1 - b_1)(q_1^2 + a_2 q_1 - b_2) \qquad (49)$$

Expanding and comparing coefficients of various powers of q_1 in (49) with coefficients of similar powers of q_1 in (44) and using (48) we get :

$$
\begin{aligned}
a_1 &= y^{1/2} \\
a_2 &= -y^{1/2} \\
b_1 &= \frac{A - y^{3/2}}{2\,y^{1/2}} \\
b_2 &= -\left(\frac{A + y^{3/2}}{2\,y^{1/2}}\right)
\end{aligned}
\qquad (50)
$$

The integral I_2 can now be expressed as

$$I_2 = \lambda \int_{U_1}^{U_2} \frac{dq_1}{(q_1^2 + a_1 q_1 - b_1)\,(q_1^2 + a_2 q_1 - b_2)}$$

$$= \lambda \int_{U_1}^{U_2} \left[\frac{M_1\, q_1 + N_1}{(q_1^2 + a_1 q_1 - b_1)} + \frac{M_2\, q_1 + N_2}{(q_1^2 + a_2 q_1 - b_2)} \right] dq_1 \qquad (51)$$

The values of various parameters are obtained by the technique of partial fractions, and they are calculated to be

$$M_1 = \frac{2a_1^3}{A^2 + a_1^6} = \frac{2y^{3/2}}{A^2 + y^3}$$

$$M_2 = \frac{2a_2^3}{A^2 + a_2^6} = -\frac{2y^{3/2}}{A^2 + y^3}$$

$$N_1 = \frac{A + 2a_1^3}{A^2 + a_1^6} \cdot a_1 = \left(\frac{A + 2y^{3/2}}{A^2 + y^3}\right) y^{1/2}$$

and

$$N_2 = \frac{A + 2a_2^3}{A^2 + a_2^6} \cdot a_2 = -\left(\frac{A - 2y^{3/2}}{A^2 + y^3}\right) \cdot y^{1/2}$$

(52)

Now $\displaystyle\int \frac{M_1 q_1 + N_1}{q_1^2 + a_1 q_1 - b_1}\, dq_1$

$$= \int \frac{M_1}{2}\, \frac{\left[2q_1 + a_1 - a_1 + \frac{2N_1}{M_1}\right] dq_1}{q_1^2 + a_1 q_1 - b_1}$$

$$= \frac{M_1}{2} \int \frac{2q_1 + a_1}{q_1^2 + a_1 q_1 - b_1}\, dq_1 + \left(N_1 - \frac{a_1 M_1}{2}\right)$$

$$\int \frac{dq_1}{\left(q_1 + \frac{a_1}{2}\right)^2 - \left(b_1 + \frac{a_1^2}{4}\right)}$$

$$= \frac{y^{3/2}}{A^2 + y^3} \ln\left(q_1^2 + a_1 q_1 - b_1\right) + \frac{\left(A + y^{3/2}\right) y^{1/2}}{A^2 + y^3}$$

$$\int \left[\frac{d\, q_1}{\left(q_1 + \frac{a_1}{2}\right) - \frac{\sqrt{\Delta_1}}{2}} - \frac{dq_1}{\left(q_1 + \frac{a_1}{2}\right) + \frac{\sqrt{\Delta_1}}{2}}\right]$$

where

$$\left(\frac{A + y^{3/2}}{A^2 + y^3}\right) y^{1/2} = N_1 - \frac{a_1 M_1}{2}$$

$$\frac{\Delta_1}{4} = b_1 + \frac{a_1^2}{4} = \frac{2A - y^{3/2}}{4\, y^{1/2}}$$

and $\Delta_1 = \dfrac{2A - y^{3/2}}{y^{1/2}} \neq 0$

$$\therefore \quad \int_{U_1}^{U_2} \frac{M_1 q_1 + N_1}{q_1^2 + a_1 q_1 - b_1}\, d\, q_1$$

$$= \frac{y^{3/2}}{A^2 + y^3} L_n \frac{U_2^2 + U_2\, y^{1/2} - \left(\dfrac{A - y^{3/2}}{2\, y^{1/2}}\right)}{U_1^2 + U_1\, y^{1/2} - \left(\dfrac{A - y^{3/2}}{2\, y^{1/2}}\right)}$$

$$+ \frac{A + y^{3/2}}{A^2 + y^3} \cdot \frac{y}{(\Delta_1\, y)^{1/2}} \cdot L_n$$

$$\frac{\left[2U_2 + y^{1/2} - (\Delta_1)^{1/2}\right]\left[2U_1 + y^{1/2} - (\Delta_1)^{1/2}\right]}{\left[2U_2 + y^{1/2} + (\Delta_1)^{1/2}\right]\left[2U_1 + y^{1/2} - (\Delta_1)^{1/2}\right]} \qquad (53)$$

Similarly, we can evaluate the second half of integral I_2

$$\int \frac{M_2 q_1 + N_2}{q_1^2 + a_2 q_1 - b_2}\, dq_1 = \frac{M_2}{2} \int \frac{\left(2q_1 + a_2 - a_2 + \dfrac{2N_2}{M_2}\right) dq_1}{\left(q_1 + \dfrac{a_2}{2}\right)^2 - \left(b_2 + \dfrac{a_2^2}{4}\right)}$$

$$= \frac{M_2}{2}\int \frac{2q_1 + a_2}{q_1^2 + a_2 q_1 - b_2}\, dq_1 + \left(N_2 - \frac{a_2 M_2}{2}\right)\int \frac{dq_1}{\left(q_1 + \frac{a_2}{2}\right)^2 - \left(\frac{+\Delta_2}{4}\right)}$$

$$= \frac{-\gamma^{3/2}}{A^2+\gamma^3} \cdot L_n\left(q_1^2 + a_2 q_1 - b_2\right) - \frac{A-\gamma^{3/2}}{A^2+\gamma^3} \cdot \gamma^{1/2}\int \frac{dq_1}{\left(q_1 + \frac{a_2}{2}\right)^2 - \left(\frac{+\Delta_2}{4}\right)}$$

$$= \frac{-\gamma^{3/2}}{A^2+\gamma^3} \cdot L_n\left(q_1^2 + a_2 q_1 - b_2\right)$$

$$- \frac{A-\gamma^{3/2}}{A^2+\gamma^3} \cdot \gamma^{1/2} \cdot \frac{2}{\sqrt{-\Delta_2}}\ \mathrm{Atan}\ \frac{q_1 + \frac{a_2}{2}}{\sqrt{\frac{-\Delta_2}{4}}}$$

where

$$\Delta_2 = b_2 + \frac{a_2^2}{4}$$

$$= -\frac{2A + \gamma^{3/2}}{4\gamma^{1/2}} \leqslant 0$$

Therefore

$$\int_{U_1}^{U_2} \frac{M_2 q_1 + N_2}{q_1^2 + a_2 q_1 - b_2} \cdot dq_1$$

$$= -\frac{\gamma^{3/2}}{A^2+\gamma^3} \cdot L_n \frac{U_2^2 - \gamma^{1/2} \cdot U_2 + \frac{A+\gamma^{3/2}}{2\gamma^{1/2}}}{U_1^2 - \gamma^{1/2} \cdot U_1 + \frac{A+\gamma^{3/2}}{2\gamma^{1/2}}}$$

$$-\frac{2(A-y^{3/2})y}{(A^2+y^3)(-\Delta_2 y)^{1/2}}\cdot \mathrm{Atan}\,\frac{(U_s-U_1)\cdot\sqrt{-\Delta_2 y}}{\left[2U_s+U_1+y\right]y^{1/2}-y(U_s+U_1)+A} \tag{54}$$

The sum of equation (53) and (54) will give the complete solution to the integral I_2.

Thus the two integrals I_1 and I_2 of equation (40) of the previous section have been evaluated exactly to enable us finding the temperature distribution in the plate as well as the surface temperature-time history for heat transfer due to combined boundary condition of Newton's law of convection and Stefan Boltzman law of radiation for the surface of the plate. The results are presented graphically in Fig. (1,2,3).

CASE II ($N_{Bi}=0$, i.e., only radiation boundary condition)

For $N_{Bi}=0$, that is for heat transfer by radiation only the Lagrangian equation (38) is reduced to the form

$$21\left[N_{Rh}\cdot q_1^N-(h+g_0''')\right]+\left[21+8\cdot N\cdot N_{Rh}\,q_1^{N-1}\right]\frac{dq_1}{dt}=0$$

where $h = N_{Rh}\cdot U_1^4$

$$\text{or}\quad \frac{21+8\,N\cdot N_{Rh}\cdot q_1^{N-1}}{21(N_{Rh}\cdot q_1^N-(h+g_0'''))}\,dq_1=-dt$$

$$\text{or}\quad \frac{N\,q_1^{N-1}+\frac{21}{8\,N_{Rh}}}{q_1^N-\frac{h+g_0'''}{N_{Rh}}}\cdot dq_1=-\frac{21}{8}\,dt \tag{55}$$

Integrating equation (55) for N = 4.0, (Stefan Boltzman's law) we have

$$\int_{U_1}^{U_s} \frac{4q_1^3}{q_1^4 - B} \cdot dq_1 + \frac{21}{8 \cdot N_{Rh}} \int_{U_1}^{U_s} \frac{dq_1}{q_1^4 - B} = -\frac{21}{8} t \qquad (56)$$

Introducing

$$z = \frac{q_1}{B^{1/4}}, \quad dz = \frac{dq_1}{B^{1/4}}$$

Equation (56) becomes

$$\int_{Z_1}^{Z_s} \frac{4z^3}{z^4 - 1} \cdot dz + \frac{21}{8\, N_{Rh}} \cdot B^{-3/4} \int \frac{dz}{z^4 - 1} = -\frac{21}{8} t$$

$$\text{or} \quad I_1 + I_2 = -\frac{21}{8} t \qquad (57)$$

where

$$I_1 = \int_{Z_1}^{Z_s} \frac{4z^3}{z^4 - 1} \cdot dz = L_n \left(\frac{Z_s^4 - 1}{Z_1^4 - 1} \right) \qquad (58)$$

and

$$I_2 = \frac{21}{8\, N_{Rh}} \cdot B^{-3/4} \int_{Z_1}^{Z_s} \frac{dz}{z^4 - 1}$$

$$= \frac{21}{8\, N_{Rh}} \cdot B^{-3/4} \int_{Z_1}^{Z_s} \left(-\frac{1}{2}\right) \left[\frac{1}{z^2 + 1} + \frac{1}{1 - z^2} \right] dz$$

$$= \frac{21}{32 N_{Rh}} \cdot B^{-3/4} \left[L_n \frac{(1+Z_s) \cdot (1 - Z_1)}{(1+Z_1)(1-Z_s)} + 2 \text{ Atan} \frac{Z_s - Z_1}{1 + Z_s Z_1} \right] \qquad (59)$$

The surface temperature given by equation (57) under present boundary condition is presented graphically in Fig. (2).

CASE III ($N_{Rh}=0$, i.e. only convection boundary condition)

For heat transfer due to Newton's law of cooling only (when $N_{Rh}=0$) the Lagrangian equation (38) becomes

$$21\left[N_{Bi}\, q_1 - (h + g_o''')\right] + \left[21 + 8\cdot N_{Bi}\right]\frac{dq_1}{dt} = 0 \qquad (60)$$

where $h = N_{Bi}\cdot U_1$ (61)

and equation (60) reduces to

$$\frac{21 + 8\cdot N_{Bi}}{21\cdot N_{Bi}}\cdot\frac{dq_1}{q_1 - U_1 - \frac{g_o'''}{N_{Bi}}} = -dt \qquad (62)$$

Integrating equation (62), we get

$$\int_{U_1}^{U_s}\frac{dq_1}{q_1 - U_1 - \frac{g_o'''}{N_{Bi}}} = -\int\omega\, dt$$

$$= L_n\frac{U_s - U_1 - \frac{g_o'''}{N_{Bi}}}{-\frac{g_o'''}{N_{Bi}}} = -\omega t \qquad (63)$$

where

$$\omega = \frac{21\cdot N_{Bi}}{21 + 8\, N_{Bi}}$$

Rearranging equation (63), we get

$$U_s = U_1 + \frac{g_o'''}{N_{Bi}}\left(1 - e^{-\omega t}\right) \qquad (64)$$

The parabolic temperature distribution (22) is now given by

$$U = \frac{N_{Bi}}{2}\left(q_1 - U_i\right)\left(1 - x^2\right) + q_1 \tag{65}$$

where equation (36) has been used to represent q_2 in terms of q_1. The surface temperature q_1 is nothing but U_s obtained in (64). Putting the values of U_s from equation (64) in equation (65) we get

$$U = \frac{g_o'''}{N_{Bi}}\left(1 - e^{-\omega t}\right)\left[\frac{N_{Bi}}{2}\left(1 - x^2\right) + 1\right] + U_i \tag{66}$$

The foregoing solution can be compared with the exact one for a body initially at zero temperature having step internal heat generation and with convective heat transfer boundary condition. The fluid temperature is also zero. The exact solution can be expressed in the following form

$$U = \frac{g_o'''}{N_{Bi}}\left[\frac{N_{Bi}}{2}\cdot\left(1 - x^2\right) + 1\right]\cdot$$

$$\left[1 - \frac{2N_{Bi}^2}{\frac{N_{Bi}}{2}\left(1 - x^2\right)+1}\cdot\sum_{n=1}^{\infty} a_o(x\,Y_n)\cdot b_o(Y_n)\,e^{-Y_n^2 t}\right] \tag{67}$$

where

$$a_o\left(xY_n\right) = \mathrm{Cos}\left(xY_n\right)$$

and $$b_o\left(Y_n\right) = \frac{1}{Y_n^2\cdot\left[Y_n^2 + N_{Bi}\left(N_{Bi}+1\right)\right]\cdot \mathrm{Cos}\,Y_n}$$

Y_n being the positive roots of the equation

$$Y \tan Y = N_{Bi} \tag{68}$$

Comparing equations (66) and (67), we notice that the expression

$$P = \frac{2 N_{Bi}^2}{\frac{N_{Bi}}{2}(1-x^2)+1} \cdot \sum_{n=1}^{\infty} a_o(xY_n) \cdot b_o(Y_n) e^{-Y_n^2 t} \qquad (69)$$

in equation (67) which is a function of time and space is replaced by an independent of space exponential function

$$P = e^{-\omega t} \qquad (70)$$

in our solution (66).

The transcenental equation (68) has infinite roots Y_n. We have calculated only six roots because the contribution of further terms is not significant while evaluating P in (69). The expression (69) is plotted graphically (FIG, 5) for the case of insulated face (x=0) and for the surface (x=1) of the plate. The results are compared with the approximate value of P given by equation (70). It is clear that the difference in the values is insignificant and hence the temperature distribution obtained by the Biot's method is comparable satisfactorily with the exact solution.

CHAPTER IV

DISCUSSION OF RESULTS

In the non-dimensional form, the unsteady heating of the plate is a three parameter problem, namely, radiation number (N_{Rh}), Biot number (N_{Bi}), and the non-dimensional initial temperature (Θ). The solution of the problem for the set of these three parameters is an undertaking of considerable magnitude. Therefore, only a few representative calculations have been performed. The problem has been solved for the following specific cases :

(I) Convection and radiation are of equal magnitude, that is, $N_{Rh}=N_{Bi}=1$, and the value of Θ is varying.

(II) The heat transfer due to convection or radiation is fixed while that due to other mode is varied over a wide range. The value of Θ is also kept constant over this range. In other words, the following cases are considered under this head -

a) $N_{Rh}=1$, Θ = constant
N_{Bi} varying

b) $N_{Bi}=1$, Θ = constant,
N_{Rh} varying

(III) Pure convection boundary condition i.e.,

$N_{Rh} = 0, \; \theta = \text{const.}$

$N_{Bi} =$ varying.

(IV) Pure radiation boundary condition i.e.,

$N_{Bi} = 0, \; \theta = \text{const.}$

N_{Rh} varying.

Taking the first case where $N_{Rh} = N_{Bi} = 1$, we make use of the equation (40) to calculate the surface temperature θ_1 as function of time t. The surface temperature has been chosen in the form

$$\frac{T_s - T_i}{T_a - T_i} = \frac{\frac{T_s}{T_a} - \frac{T_i}{T_a}}{1 - \frac{T_i}{T_a}} = \frac{\theta_1 - \theta}{1 - \theta}$$

The graph has been plotted between $\frac{T_s - T_i}{T_a - T_i}$ and the non-dimensional time t for various values of θ after evaluating integrals I_1 and I_2 discussed in equations (41) and (42). It can be seen from figure 5 that at large times the trend of the surface temperatures for varying θ, is quite different than at smaller times. All the curves at large times behave asymptotically and tend to attain the value unity at steady state. The figure also shows the variation of surface temperature for $\theta = 0$. Though physically the initial temperature cannot be equal to zero, in many situations the temperature ratio θ can be very

small. It is also seen from the figure that, as the dimensionless initial temperature is increased, the time needed to obtain a certain surface temperature is decreased which is true physically also.

Considering the second case ($N_{Rh} = 1$, θ = const, N_{Bi} varying) and ($N_{Bi} = 1$, θ = const, N_{Rh} varying), we have plotted the results in figure (2) and (3) for surface temperature as function of time. Again the same expressions (40), (41), (42) have been used. The results are for the case when $\theta = 0.4$. As expected, pure convection and pure radiation cases act as lower bounds for the combined radiation and convection solutions. An increase in one of the two parameters results in a decrease in the time required to reach a given dimensionless temperature. This is clear from both the figures (2) and (3).

The results for pure radiation boundary condition and pure convection condition have not been illustrated graphically separately because these results already occur in figures (2) and (3) as limiting cases. In order to however, compare our results with the exact solution and to test the accuracy of the Biot's variational approach we have plotted our approximate solution (66) and the exact solution (67) for the temperature distribution at surfaces $x = 0$, and at $x = 1$ for the pure convection case. Figure(4) is used to choose values of the function P explained in (69) and (70) at different times. Figure (5) is a comparison of the Biot's and exact solution showing very good agreement.

Conclusion :

Biot's variational principle has been extended to heat conduction problem with combined convection and radiation non-linear boundary condition. Satisfactory results are obtained as compared to the exact solution. The mathematical simplicity achieved, helps to overcome the complexity of the problem which has been experienced with exact analysis.

The Biot's method may be further extended to other transient heat conduction problems, for example, those with the heat transfer coefficient a function of the surface temperature and/or time. It may also be applied to other one-dimensional geometries such as the infinite circular cylinder, circular plate and the sphere.

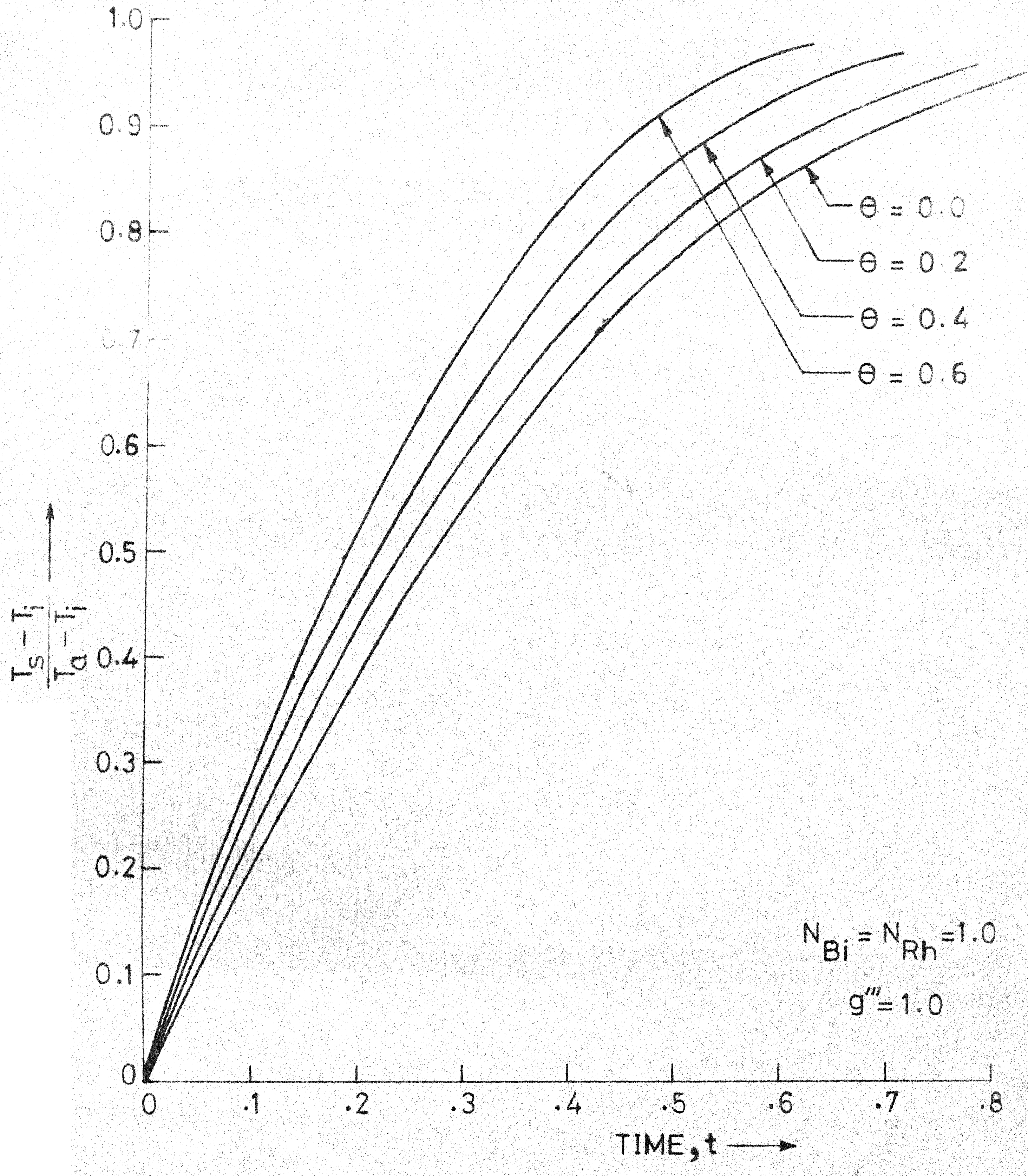


FIG 1 EFFECT OF INITIAL TEMP. DISTRIBUTION θ ON SURFACE TEMP.

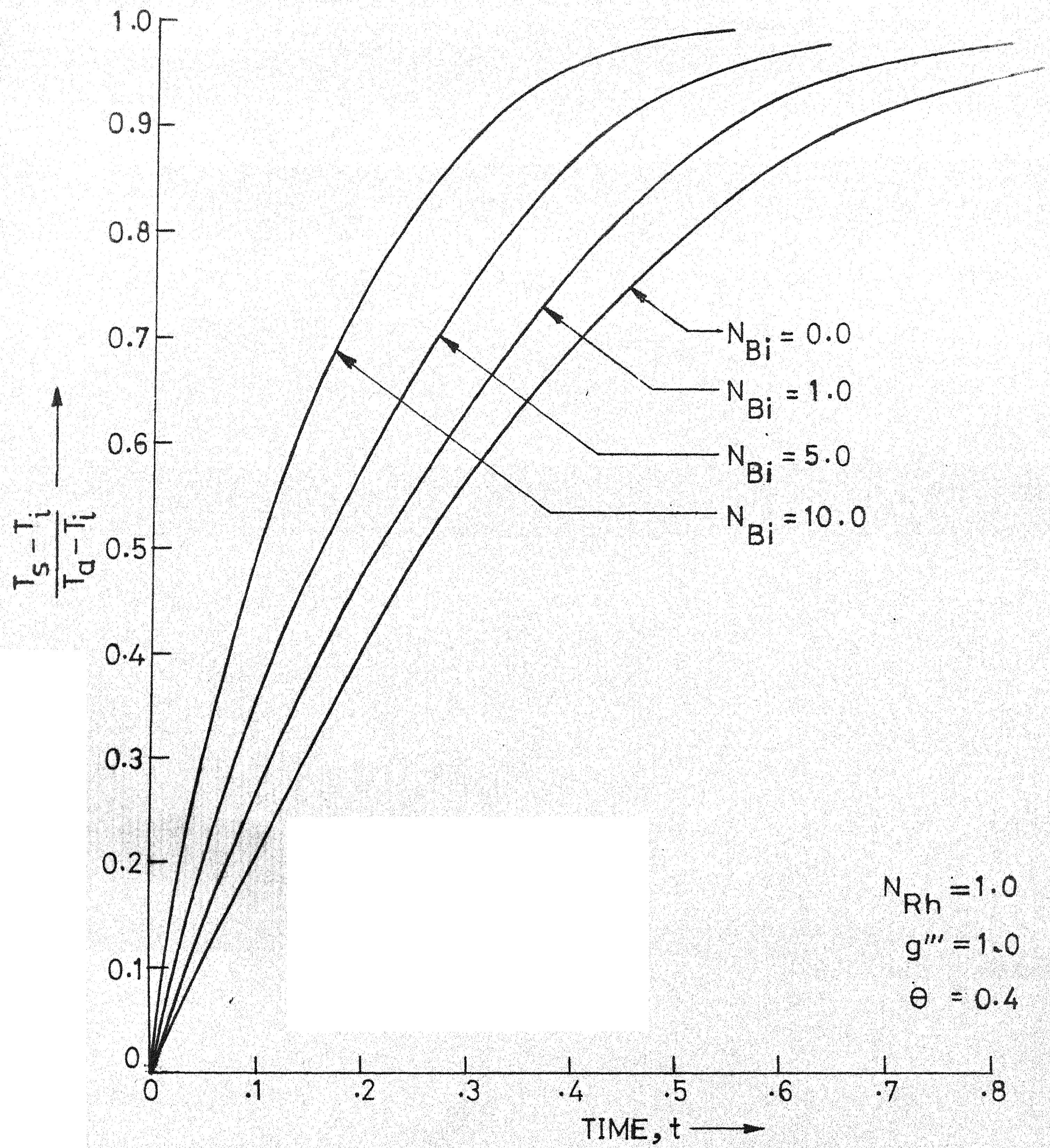


FIG. 2 EFFECT OF BOIT NUMBER ON SURFACE TEMP.

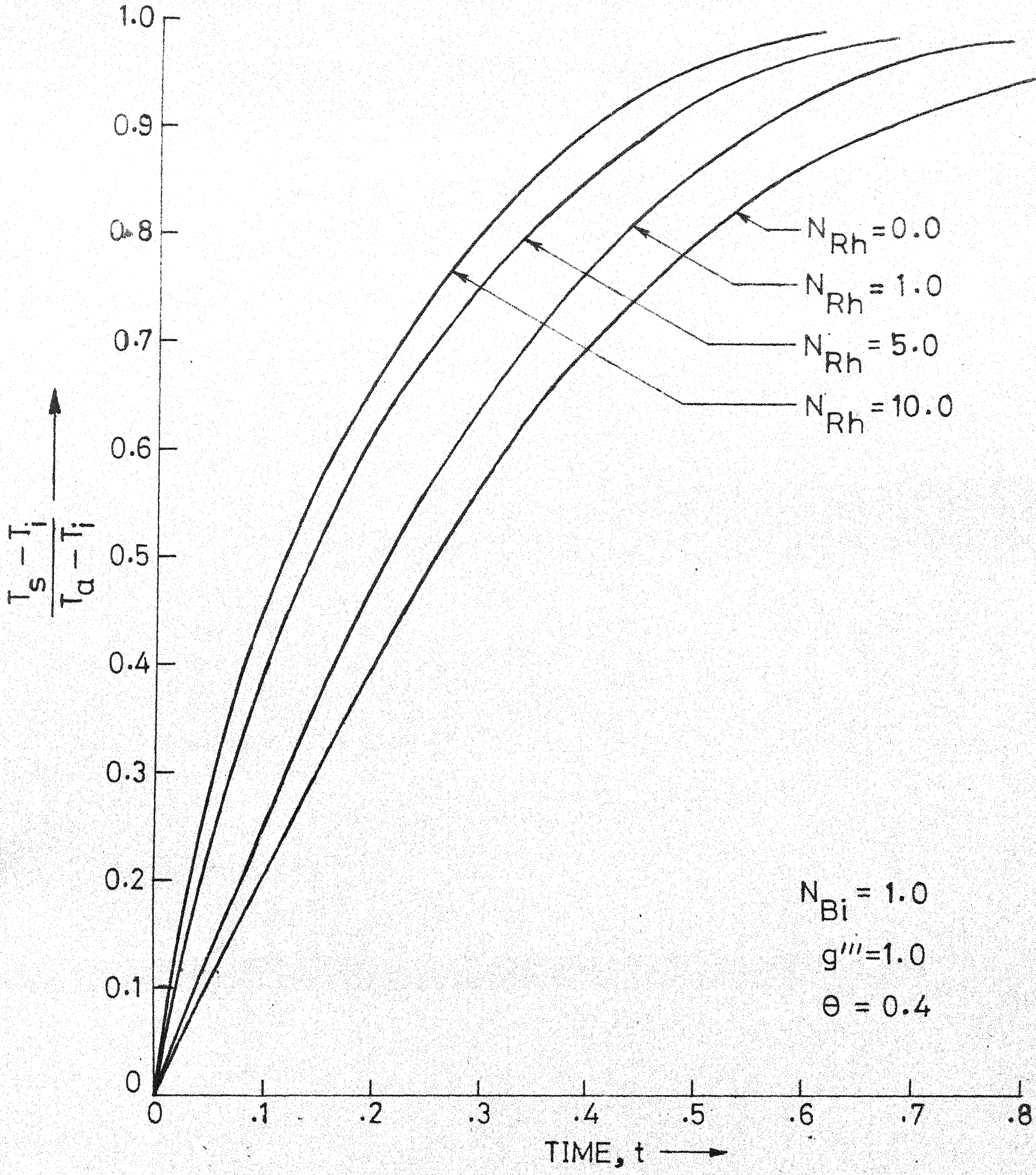


FIG. 3 EFFECT OF RADIATION NUMBER ON SURFACE TEMP.

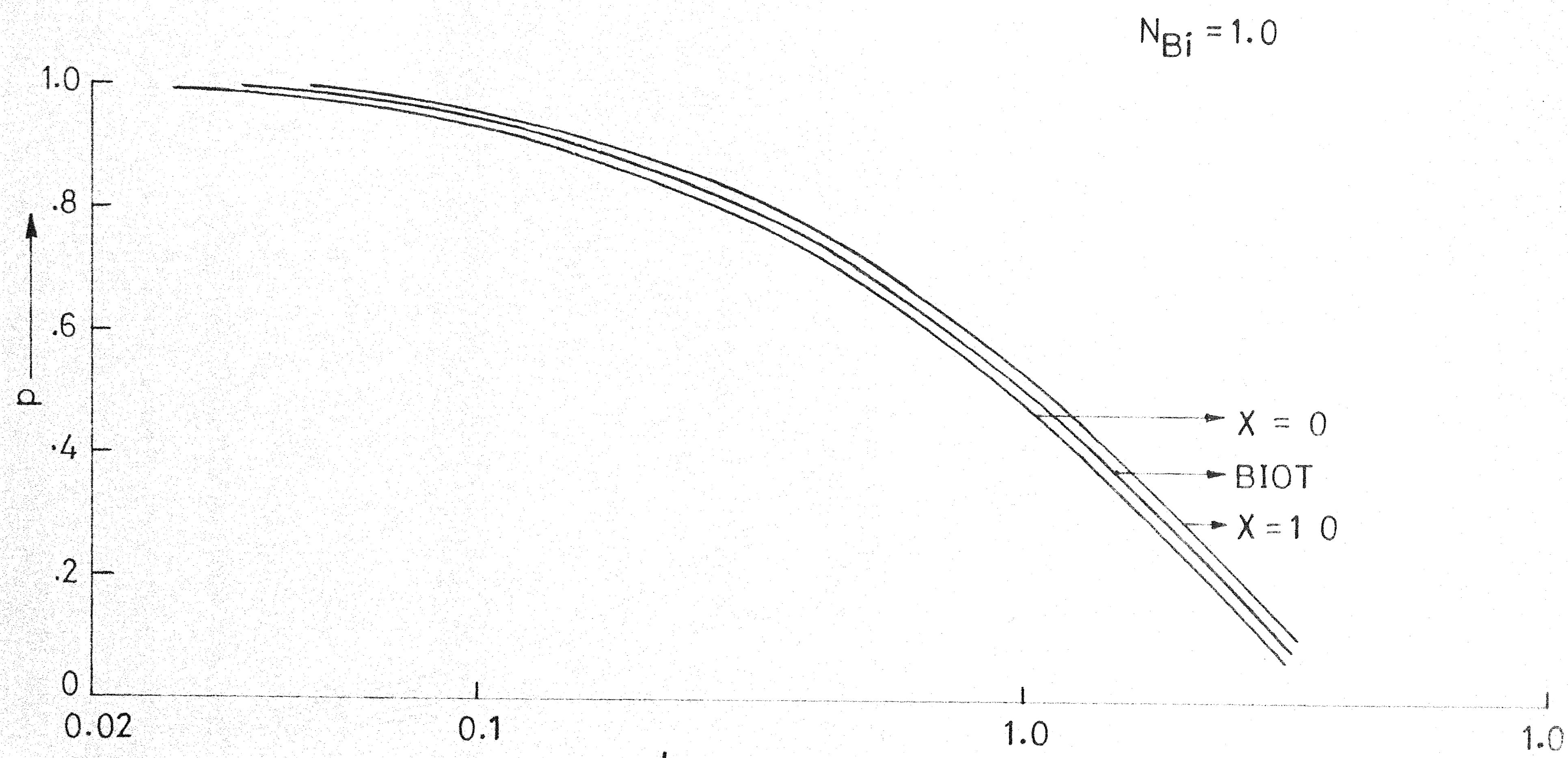


FIGURE 4

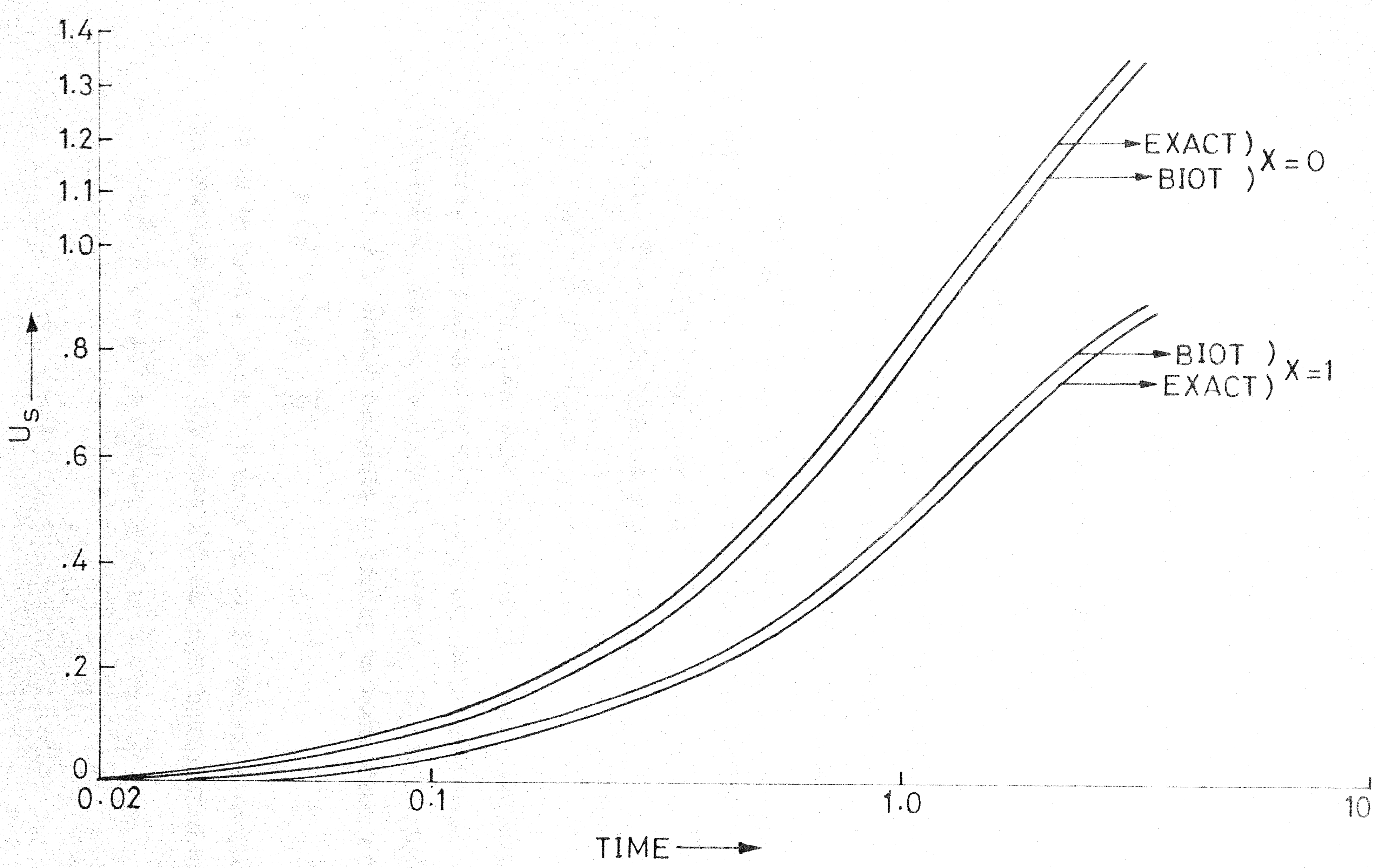

1.4
1.3
1.2
1.1
1.0
.8
.6
.4
.2
0
U_s
0.02
0.1
1.0
10
TIME
EXACT)
BIOT)
X = 0
BIOT)
EXACT)
X = 1

FIGURE 5

APPENDIX I

SOLUTION OF BIQUADRATIC EQUATIONS

FERRARI'S METHOD

The algebraic solution of biquadratic equations was discovered by Ferrari, a pupil of Cardan.

Let the biquadratic equation be given by

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

or

$$x^4 + ax^3 = - bx^2 - cx - d$$

Adding $\frac{a^2}{4} \cdot x^2$ to both sides, we obtain

$$(x^2 + \frac{a}{2} x)^2 = (\frac{a^2}{4} - b)\, x^2 - cx - d \qquad (1)$$

If the right hand member of (1) were a perfect square, the solution of this equation would have been immediate. But in general it is not so. The basic idea in Ferrari's method consists in adding to both sides of (1)

$$y\,(x^2 + \frac{a}{2} x) + \frac{y^2}{4}$$

so as to have a perfect square in the left-hand side for an indeterminate y. Equation (1) is then transformed into

$$(x^2 + \frac{a}{2} x + \frac{y}{2})^2 = (\frac{a^2}{4} - b + y)\, x^2 + (-c + \frac{1}{2} ay)x$$
$$+ (- d + \frac{1}{4} y^2) \qquad (2)$$

Now we seek to determine y so that

$$\left(\frac{a^2}{4} - b + y\right) x^2 + \left(-c + \frac{1}{2}\, ay\right) x + \left(-d + \frac{1}{4}\, y^2\right) \tag{3}$$

becomes the square of a linear expression $(e\,x + f)$.

In general, if

$$A\,x^2 + B\,x + C = (e\,x + f)^2 \tag{4}$$

then

$$B^2 - 4AC = 0 \tag{5}$$

and conversely. In fact, equation (4) is equivalent to the three relations

$$A = e^2\,, \qquad B = 2\,ef, \qquad C = f^2 \tag{6}$$

so that the condition (5) is satisfied. Conversely, suppose that (5) holds. Then, if both $A = 0$ and $C = 0$, we have also $B = 0$, and the relation (6) will hold for $e = f = 0$. If both A and C are not zero let, for example, $A \neq 0$. Then, we take

$$e = \sqrt{A}\,, \qquad f = \frac{B}{2e}\,,$$

and by virtue of (5) we shall have

$$C = f^2$$

Thus, the right-hand side of (2) will be the square of a linear expression $(e\,x + f)$ if y satisfies the equation

$$\left(\frac{1}{2}\, ay - c\right)^2 = 4\left(y + \frac{a^2}{4} - b\right)\left(\frac{1}{4}\, y^2 - d\right)$$

or in the expanded form,

$$y^3 - by^2 + (ac - 4d)\,y + 4bd - a^2d - c^2 = 0 \tag{7}$$

It suffices to take for y any root of this cubic equation Appendix II , called the resolvent of the biquadratic equation, in order to have

$$\left(\frac{a^2}{4} - b + y\right)x^2 + \left(\frac{1}{2} ay - c\right)x + \frac{1}{4} y^2 - d = (ex + f)^2$$

with properly chosen e and f. The biquadratic equation(2) appears then in the form

$$\left(x^2 + \frac{a}{2} x + \frac{1}{2} y\right)^2 = (ex + f)^2$$

and splits into two quadratic equations

$$x^2 + \frac{a}{2} x + \frac{1}{2} y = e x + f$$

$$x^2 + \frac{a}{2} x + \frac{1}{2} y = -e x - f$$

The solution of the above equations is simple and well known. This gives the required four roots of the biquadratic equation.

APPENDIX II

SOLUTION OF CUBIC EQUATION

CARDAN'S FORMULA

The general cubic equation takes the form

$$f(x) = x^3 + ax^2 + bx + c = 0 \qquad (1)$$

By introducing a new unknown this equation can be simplified, such that it will not contain the second power of the unknown. Let this new unknown be

$$(y + k) = x$$

with k still arbitrary. By Taylor's formula

$$f(y + k) = f(k) + f'(k)\ y + \frac{f''(k)}{2} \cdot y^2 + \frac{f'''(k)}{6} y^3 \qquad (2)$$

where

$$\begin{aligned} f(k) &= k^3 + ak^2 + bk + c \\ f'(k) &= 3k^2 + 2ak + b \\ \frac{1}{2} f''(k) &= 3k + a \\ \frac{1}{6} f'''(k) &= 1 \end{aligned} \qquad (3)$$

To get rid of the term involving y^2 it suffices to choose k in such a manner that

$$3k + a = 0 \qquad \text{or} \qquad k = -\frac{a}{3}$$

Then

$$f'(-\frac{a}{3}) = b - \frac{a^2}{3}$$

$$f(-\frac{a}{3}) = c - \frac{ba}{3} + \frac{2a^3}{27}$$

Substituting

$$x = y - \frac{a}{3}$$

the proposed equation (1) is transformed into

$$y^3 + py + q = 0 \tag{4}$$

where

$$p = b - \frac{a^2}{3}, \quad q = c - \frac{ba}{3} + \frac{2a^3}{27}$$

A cubic equation of the form (4) can be solved by means of the following device. We seek to satisfy it by setting

$$y = u + v \tag{5}$$

thus introducing two unknowns u and v. On substituting this expression into (4) and arranging terms in a proper way, u and v have to satisfy the equation

$$u^3 + v^3 + (p + 3\,uv)\,(u + v) + q = 0 \tag{6}$$

with two unknowns. This problem is indeterminate unless another relation between u and v is known. For this relation we choose

$$3\,uv + p = 0$$

or $\quad u\,v = -\frac{p}{3} \qquad (7)$

Then, it follows from (6) that

$$u^3 + v^3 = -q \tag{8}$$

so that the solution of the cubic (4) can be obtained by solving the system of two equations

$$\begin{aligned} u^3 + v^3 &= -q \\ uv &= -\frac{p}{3} \end{aligned} \tag{9}$$

Taking the cube of the latter equation, we have

$$u^3 v^3 = -\frac{p^3}{27} \tag{10}$$

From equations (9) and (10), we know the sum and the product of two unknown quantities u^3 and v^3. These quantities are the roots of the quadratic equation

$$t^2 + qt - \frac{p^3}{27} = 0 \tag{11}$$

Denoting them by A and B, we have then

$$\begin{aligned} A &= -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \\ B &= -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \end{aligned} \tag{12}$$

where we are at liberty to choose the square root as we please. Now owing to the symmetry between the terms u^3 and v^3 in the system (9) we can set

$$\begin{aligned} u^3 &= A \\ v^3 &= B \end{aligned}$$

If some determined value of the cube root of A is denoted by $\sqrt[3]{A}$, the three possible values of u will be

$$u = \sqrt[3]{A}, \quad u = \omega\sqrt[3]{A}, \quad u = \omega^2 \sqrt[3]{A}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2}$$

is an imaginary cube root of unity. As to v it will also have three values.

$$v = \sqrt[3]{B}, \quad v = \omega\sqrt[3]{B}, \quad v = \omega^2\sqrt[3]{B}$$

but not every one of them can be associated with the three possible values of u, since u and v must satisfy the relation

$$u\,v = -\frac{p}{3}$$

If $\sqrt[3]{B}$ stands for that cube root of B which satisfies the relation

$$\sqrt[3]{A} \cdot \sqrt[3]{B} = -\frac{p}{3}$$

then the values of v which can be associated with

$$u = \sqrt[3]{A}, \quad u = \omega\sqrt[3]{A}, \quad u = \omega^2\sqrt[3]{A}$$

will be

$$v = \sqrt[3]{B}, \quad v = \omega^2\sqrt[3]{B}, \quad v = \omega\sqrt[3]{B}$$

Hence, equation (1) will have the following roots

$$y_1 = \sqrt[3]{A} + \sqrt[3]{B}$$

$$y_2 = \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}$$

$$y_3 = \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}$$

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